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Renormalization Group Analysis of the Global Properties of a Strange Attractor

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This paper considers the circle map at the special point: the one at which there is a trajectory with a golden mean winding number and at which the map just fails to be invertable at one point on the circle. The invariant density of this trajectory has fractal properties. Previous work has suggested that the global behavior of this fractal can be effectively analyzed using a kind of partition function formalism to generate an f versus α curve. In this paper the partition function is obtained by using a renormalization group approach.

KEY WORDS: Renormalization group; circle map; strange attractor; fractal dimension; golden mean.

1. INTRODUCTION

Several recent papers[1,2] have suggested that the global properties of fractal sets can be analyzed via a partition function approach. Specifically, the properties of the critical circle map, defined via

$$\theta_{j+1} = f(\theta_j) = \theta_j + \Omega - (\sin 2\pi\theta_j)/2\pi \tag{1.1}$$

have been analyzed in this manner. The particular case considered most fully[2] is the one in which the winding number

$$\lim_{j \to \infty} \frac{\theta_j - \theta_0}{j} = \omega(\Omega) \tag{1.2}$$

takes on a value equal to the "golden mean"

$$\omega = (\sqrt{5} - 1)/2 = -g \tag{1.3}$$

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For this case, the trajectory given by (1.1) has an invriant density on the circle, $\phi_j = \theta_j$ (mod 1), with highly nontrivial and apparently universal topological properties.

The physical nature of these properties is simply described. Consider any very long trajectory θ_j, $j = 0, 1, 2,\ldots$. On the circle, the trajectory elements have a simple and orderly set of return properties, which can be described by the Fibonacci numbers, F_n. These obey

$$\begin{aligned} F_0 &= F_1 = 1 \\ F_{n+1} &= F_n + F_{n-1} \end{aligned} \tag{1.4}$$

For high n, θ_j and θ_{j+F_n} lie very close to one another on the circle. Thus, if we define

$$u_{n,j} = \theta_{j+F_n} - \theta_j - F_{n-1} \tag{1.5}$$

we can notice that as n goes to infinity, $u_{n,j}$ goes to zero for all j. The topological properties of the trajectory are then defined by giving the distribution of values of $u_{n,j}$ for fixed n as j is varied. The range of variation of $u_{n,j}$ with j is very large. In fact, $\ln |u_{n,j}|$ has a range in j which varies linearly in n. To describe the statistical properties of $u_{n,j}$, we introduce

$$\Gamma_n(\tau) = \langle |u_{n,j}|^{-\tau} \rangle \tag{1.6}$$

In (1.6) the average is over all j, i.e., all points in the trajectory.

In Ref. 1, the dependence of Γ_n upon n was analyzed and we saw that for large n

$$\lim_{n \to \infty} \frac{\ln \Gamma_n(\tau)}{n} = [\ln |g|)(1 - q(\tau)] \tag{1.7}$$

where $q(\tau)$ is independent of n and is the essential quantity needed to describe the properties of the attractor.

The purpose of the present paper is to derive (1.7) and a set of renormalization equations appropriate for calculating $q(\tau)$.

In one sense, this derivation should be simple and straightforward. Consider the fractal objects constructed by considering not the entire set of θ_js but only the subset defined by insisting that the value of θ_j on the circle lie in some small range $[\phi_{\min}, \phi_{\max}]$. For $\phi_{\min}$ and $\phi_{\max}$ near zero, there is a renormalization group calculation which effectively describes this subset.[3,4] Hence the contribution of this region of θ_j to $\Gamma_n(\tau)$ might well be easily expressed by standard renormalization arguments. On the other hand, the mapping $f(\theta)$ takes any region of the circle into any other. Hence the fractal is in some sense homogeneous. It is fully determined by, and essentially consists of, its small ϕ behavior.

The rest of this paper is simply the working out of these ideas. The next section elaborates the well-known small ϕ scaling properties and ends with an expression for the relation between $u_{j,n}$ in different regions of the lattice. The third section describes a renormalization group theory based upon these scaling properties. The fourth gives results.

2. SCALING AND GLOBAL PROPERTIES OF THE CIRCLE MAP

A. Definitions of the Basic Quantities

The algebraic properties of the golden mean trajectory are rooted in the properties of Fibonacci numbers and of the golden mean itself.[5] Just as the Fibonacci numbers obey the recursion relation (1.4) so the golden mean, $-g$, obeys

$$g^{n+1} = g^n + g^{n-1} \tag{2.1}$$

and the two are related by

$$gF_n = -F_{n-1} + g^n \tag{2.2}$$

High iterates of the original mapping function (1.1) may be defined recursively by giving a quantity involving F_n iterations by

$$f_{n+1} = f_n \circ f_{n-1} \tag{2.3}$$

with the initial conditions

$$\begin{aligned} f_0(\theta) &= \theta - 1 \\ f_1(\theta) &= f(\theta) \end{aligned} \tag{2.4}$$

The periodicity condition $f(\theta - 1) = f(\theta) - 1$, then ensures that the entire set of f_n forms a set of functions which commutes under the recursion operation

$$f_n \circ f_m = f_m \circ f_n \tag{2.5}$$

The basic quantity $u_{n,j}$ is then given by

$$u_{n,j} = f_{n+1}(\theta_j) - \theta_j \tag{2.6}$$

The value of the bare winding number, Ω, is chosen to get the actual winding number to be the golden mean and thereby ensure that $u_{n,j}$ goes to zero as $n \to \infty$.

The properties of the orbit θ_j may be described by giving a conjugacy function[6] $\theta(t)$ which obeys

$$f[\theta(t)]=\theta(t-g) \tag{2.7a}$$

and the periodicity condition

$$\theta(t-1)=\theta(t)-1 \tag{2.7b}$$

Here $\theta(t)$ is continuous but nowhere differentiable. Equations (2.3) and (2.4) may be combined with (2.7) to define the effect of any f_n upon $\theta(t)$, namely

$$f_n[\theta(t)]=\theta(t-g_n) \tag{2.8}$$

For any initial starting point $\theta_0=\theta(t_0)$ the subsequent θ values obey $\theta_j=\theta(t_0-jg)$ and the subsequent t values are $t_j=t_0-jg$. When these are shifted by integers to lie in the interval $(0, 1]$ the shifted t values are uniformly distributed over the interval. Hence the basic generating function, $\Gamma_n(t)$ can be written as

$$\Gamma_n(\tau)=\int_{-g^2}^{-g} dt\, \frac{1}{|f_{n+1}[\theta(t)]-\theta(t)|^\tau} \tag{2.9}$$

In our later work, it will be important for us to know the positions of the singularities of $f_n(\theta)$. These singularities are zeroes in the derivatives of $df_n/d\theta$. Since $f_0(\theta)$ has a singularity at $\theta=0$, each and every f_n for $n\geqslant 1$ will have such a singularity at $\theta(0)$. Define the zero of the conjugacy function by choosing $\theta(0)=0$. From the recursion formula it follows that $f_n(\theta)$ will have singularities at all the points $\theta=\theta(jg)$, $j=0, 1, 2,..., F_{n-1}-1$. When these singularities are expressed in the interval $(-\frac{1}{2}, \frac{1}{2})$ we find that the singularities nearest to zero are at

$$\theta=0; \qquad \theta=\theta(g^{n-2}), \qquad \theta=\theta(g^{n-3}) \tag{2.10}$$

for $n\geqslant 3$.

Conversely, we can consider the inverse function $h_n(\theta)$ which obeys

$$h_n[f_n(\theta)]=\theta$$

or

$$h_n[\theta(t)]=\theta(t+g^n) \tag{2.11}$$

Since $h_n(\theta)$ has a derivative which obeys

$$\frac{dh_n(\theta)}{d\theta}=\frac{1}{(df_n(x)/dx)|_{x=h_n(\theta)}} \tag{2.12}$$

$h_n(\theta)$ has no singularities in the range of θ values given by $\theta(t)$ with

$$-1 < t/g^n < -g^{-1} \tag{2.13}$$

B. The Basic Theorem

We are now in a position to state our basic theorem. Define the quantity in the denominator of (2.9) to be

$$u_n(t) = \theta(t - g^{n+1}) - \theta(t) \tag{2.14}$$

Notice that this can be rewritten as

$$u_n(t) = f_m[\theta(t - g^{n+1} + g^m)] - f_m[\theta(t + g^m)] \tag{2.15}$$

let

$$n \gg m \gg 1 \tag{2.16}$$

Since n is so large the two arguments of the f_ms in (2.15) differ by very little. Just as long as we stay away from the singularities of f_m it is possible to expand in their difference and find

$$u_n(t) = f'_m[\theta(t + g^m)]\, u_n(t + g^m)$$

or equivalently

$$u_n(t + g^m) = u_n(t)\, h'_m[\theta(t)] \tag{2.17}$$

whenever t lies in a range bounded away from the singularities of (2.13), for example when

$$-1 < t/g^m < -g^{-1} \tag{2.18}$$

C. Scaling Behavior

All of the functions in question have a particularly simple behavior in the limit $n \to \infty$, $t \to 0$, $\theta \to 0$. In this scaling limit, there is a natural scale for t, g^n, and a natural scale for f_n and u_n, α^n. Here $\alpha = -1.28857...$ is the scale factor defined in Refs. 3 and 4. Define, using the logic of Feigenbaum's original renormalization work[7]

$$\begin{aligned} \alpha^n f_n(x\alpha^{-n}) &= \tilde{f}_n(x) \\ \alpha^n \theta(sg^n) &= \tilde{\theta}_n(s) \\ \alpha^n u_n(sg^n) &= \tilde{u}_n(s) \end{aligned} \tag{2.19}$$

Each of the quantities on the right-hand-side of (2.19) achieves a nontrivial limit as $n \to \infty$. Call these limit functions $\tilde{f}$, $\tilde{\theta}$, $\tilde{u}$. The equations above imply relations for these new functions. We now simply list the relations which will be useful in what follows: (2.3) implies recursion relations of f_n. Define $\tilde{h}_n$ and $\tilde{h}$ to be, respectively, the inverses of $\tilde{f}_n$ and $\tilde{f}$. They obey

$$\tilde{h}_{n+1}(x) = \alpha^2 \tilde{h}_{n-1}[\alpha^{-1} \tilde{h}_n(x/\alpha)] \tag{2.20a}$$

$$= \alpha \tilde{h}_n[\alpha \tilde{h}_{n-1}(x/\alpha^2)] \tag{2.20b}$$

while

$$\tilde{h}(x) = \alpha^2 \tilde{h}[\alpha^{-1} \tilde{h}(x/\alpha)] \tag{2.21a}$$

$$= \alpha \tilde{h}[\alpha \tilde{h}(x/\alpha^2)] \tag{2.21b}$$

Equation (2.19) directly implies a scaling law for the conjugacy function[8]

$$\alpha^n \tilde{\theta}(s) = \tilde{\theta}(s/g^n) \tag{2.22}$$

for all integral values of n. Equation (2.7a) then implies

$$\tilde{h}[\tilde{\theta}(s)] = \tilde{\theta}(s+1) \tag{2.23a}$$

and also the inverse relation

$$\tilde{f}[\tilde{\theta}(s)] = \tilde{\theta}(s-1) \tag{2.23b}$$

The scaling-limit displacement function $\tilde{u}(s)$ is defined via (2.14) to give

$$\begin{aligned} \tilde{u}(s) &= \tilde{\theta}(s-g) - \tilde{\theta}(s) \\ &= \alpha^{-1} \tilde{f}[\alpha \tilde{\theta}(s)] - \tilde{\theta}(s) \end{aligned} \tag{2.24}$$

Finally, the crucial statement (2.17) becomes

$$\tilde{u}(s + g^{-n}) = \tilde{u}(s)\, \tilde{h}'[\tilde{\theta}(sg^n)] \tag{2.25}$$

for $n \gg 1$. According to (2.13), $\tilde{h}(\theta)$ is nonsingular if θ lies in the range

$$\theta(-g^{-1}) > \theta > \theta(-1) \tag{2.26}$$

so that (2.25) is acceptable whenever s lies in the range

$$-g^{-1} > sg^n > -1 \tag{2.27}$$

3. THE RENORMALIZATION GROUP

A. "Exact" Statements about the Generating Function

In this section, we outline some of the exact statements of the $n \to \infty$ limit of $\Gamma_n(\tau)$. We restrict ourselves here to results which can be obtained without using any renormalization group results. We write

$$\begin{aligned}
\Gamma_n(\tau) &= \int_{-g^2}^{-g} dt \frac{1}{|u_n(t)|^\tau} \\
&= \int_{-g^2}^{-g} dt \frac{1}{|\theta(t - g^{n+1}) - \theta(t)|^\tau} \\
&= A_n(\tau)\, |g|^{-n[q(\tau)-1]}
\end{aligned} \tag{3.1}$$

Here $A_n(\tau)$ varies with less than exponential rapidity in n as $n \to \infty$. What can we say about $q(\tau)$?

At $\tau = 0$, the integral is trivial and we find at once that

$$q(0) = 1 \tag{3.2}$$

At $\tau = -1$, the integral reduces to

$$\begin{aligned}
\Gamma_n(-1) &= \left[\int_{-g^2-g^{n+1}}^{-g-g^{n+1}} dt\, \theta(t) - \int_{-g^2}^{-g} dt\, \theta(t)\right](-1)^n \\
&= \left[\int_{-g^2}^{-g^2-g^{n+1}} dt\, \theta(t) - \int_{-g}^{-g-g^{n+1}} dt\, \theta(t)\right](-1)^{n+1} \\
&= \int_{-g}^{-g-g^{n+1}} dt\, [\theta(t) - \theta(t-1)](-1)^n
\end{aligned}$$

Since $\theta(t-1) = \theta(t) - 1$, we find

$$\Gamma_n(-1) = |g|^{n+1}$$

and hence obtain another value of q

$$q(-1) = 0 \tag{3.3}$$

As $\tau \to -\infty$, the integrals in (3.1) pick out the very largest values of $u_n(t)$. These maxima occur near $t = 0$ where $|u_n(t)| \sim |\alpha|^{-n}$ and occur over a range $|t| \sim |g|^n$. This estimate gives

$$\Gamma_n(\tau) \sim |\alpha|^{n\tau}\, |g|^n$$

or

$$q(\tau) \to -\tau(\ln|\alpha|/\ln|g|) \tag{3.4}$$

as $\tau \to -\infty$. Correspondingly as $\tau \to \infty$, the integral picks out the smallest values of $u_n(t)$. Since

$$u_n(t-g) = f[\theta(t-g^{n+1})] - f[\theta(t)]$$

and since $f(\theta)$ has a cubic inflection point at $\theta = 0$, we find that

$$|u_n(t-g)| \sim |\alpha|^{-3n} \qquad \text{for} \qquad |tg^{-n}| \sim 1$$

In this way we obtain the large τ estimate

$$q(\tau) \to -\tau(3 \ln|\alpha|/\ln|g|) \qquad \text{as} \qquad \tau \to \infty \tag{3.5}$$

B. Formulation

Define the basic quantity in the renormalization analysis

$$\Gamma_{n,m}(\tau, [\psi]) = (-1)^m \int_{-g^{m+1}}^{-g^{m+2}} dt \frac{\psi[\theta(g^{-m}t)]}{|U_n(t)|^\tau} \tag{3.6}$$

$\Gamma_{n,m}$ is thus a functional of the function $\psi(\theta)$. We take this function to be smooth and nonsingular in the interval

$$\theta(g) < \theta < \theta(g^2) \tag{3.7}$$

The quantity which we really wish to know is

$$\Gamma_n(\tau) = \Gamma_{n,o}(\tau, [1]) \tag{3.8}$$

where [1] stands for the function $\psi(\theta)$ which is always unity

$$1(\theta) = 1$$

In the scaling limit

$$n \gg m \gg 1 \tag{3.9}$$

$\Gamma_{n,m}$ reduces to

$$\Gamma_{n,m}(\tau, [\psi]) = |\alpha|^{n\tau} |g|^n \gamma_{n-m}(\tau, [\psi]) \tag{3.10}$$

where the scaling function is

$$\gamma_n(\tau, [\psi]) = (-1)^n \int_{-g^{-n+2}}^{-g^{-n+1}} ds \frac{\psi[\tilde{\theta}(g^n s)]}{|\tilde{u}(s)|^\tau} \tag{3.11}$$

Our analysis involves splitting the integral into two parts

$$\gamma_n(\tau, [\psi]) = I_1 + I_2$$
$$I_1 = (-1)^n \int_{-g^{-n+2}}^{-g^{-n+3}} ds \frac{\psi[\tilde{\theta}(g^n s)]}{|\tilde{u}(s)|^\tau} \tag{3.12}$$
$$I_2 = (-1)^n \int_{-g^{-n+3}}^{-g^{-n+1}} ds \frac{\psi[\tilde{\theta}(g^n s)]}{|\tilde{u}(s)|^\tau}$$

and then showing that as $n \to \infty$, I_1, and I_2 can be, respectively, written as

$$\begin{aligned} I_1 &= \gamma_{n-1}(\tau, [\psi_1]) \\ I_2 &= \gamma_{n-2}(\tau, [\psi_2]) \end{aligned} \tag{3.13}$$

where ψ_1 and ψ_2 are functional of θ simply related to the original function ψ. This relationship is expressed by writing

$$\psi_j = G_j[\psi] \qquad \text{for} \quad j = 1, 2 \tag{3.14}$$

The functional relationship turns out to be very simple indeed. First, notice that I_1 may be written as

$$I_1 = (-1)^{n-1} \int_{-g^{-(n-1)+2}}^{-g^{-(n-1)+1}} ds \frac{\psi[\alpha^{-1}\tilde{\theta}(g^{n-1}s)]}{|\tilde{u}(s)|^\tau}$$

Therefore the new function is

$$\psi_1(\theta) = \psi(\alpha^{-1}\theta) \tag{3.15a}$$

or equivalently expressed as

$$G_1[\psi] = \psi \circ \alpha^{-1} \tag{3.15b}$$

where α^{-1} is the function $\alpha^{-1}(\theta) = \theta/\alpha$.

The calculation of G_2 is only slightly more complex. Look at the second integral in (3.12). Whenever s appears, write instead $s = s + g^{-n+2}$. Then the integral becomes

$$I_2 = (-1)^n \int_{-g^{-n+3} - g^{-n+2}}^{-g^{-n+1} - g^{-n+2}} ds \frac{\psi[\tilde{\theta}(g^n s + g^2)]}{|\tilde{u}(s + g^{2-n})|^\tau} \tag{3.16}$$

The upper and lower limits in (3.16) are, respectively, $-g^{-n+3}$ and $-g^{-n+4}$. Equation (2.25) enables us to replace $\tilde{u}(s + g^{2-n})$ for large n by $\tilde{u}(s)\, \tilde{h}'[\tilde{\theta}(g^{n-2}s)]$. Equations (2.19) and (2.23b) imply

$$\begin{aligned} \tilde{\theta}(g^n s + g^2) &= \alpha^{-2}\tilde{\theta}(g^{n-2}s + 1) \\ &= \alpha^{-2}\tilde{h}[\tilde{\theta}(g^{n-2}s)] \end{aligned}$$

Hence, the integral (3.15) may be rewritten as

$$I_2 = (-1)^{(n-2)+1} \int_{-g^{-(n-2)+2}}^{-g^{-(n-2)+1}} ds \left. \frac{\psi_2(\theta)}{|\tilde{u}(\theta)|^{\tau}} \right|_{\theta = \tilde{\theta}(g^{2-n}s)} \tag{3.17}$$

Hence we obtain the second part of (3.13) with

$$\psi_2(\theta) = \frac{\psi[\alpha^{-2}\tilde{h}(\theta)]}{[\tilde{h}'(\theta)]^{\tau}} \tag{3.18a}$$

In functional language (3.18a) can be written as

$$\begin{aligned} \psi_2 &= G_2[\psi] \\ G_2[\psi] &= \frac{\psi \circ \alpha^{-1} \circ \alpha^{-1} \circ \tilde{h}}{[\tilde{h}']^{\tau}} \end{aligned} \tag{3.18b}$$

Our renormalization group equation can thus be stated as

$$\gamma_n(\tau, [\psi]) = \gamma_{n-1}\{\tau, (G_1[\psi])\} + \gamma_{n-2}\{\tau, (G_2[\psi])\} \tag{3.19}$$

where the functionals G_1 and G_2 are defined, respectively, by (3.15) and (3.18). The full specification of (3.19) must include a statement of the domain of definition of ψ. Equation (3.11) states that the domain of $\psi(\theta)$ is

$$\theta(-g^2) \leqslant \theta \leqslant \theta(-g) \tag{3.20a}$$

Then in $G_1[\psi](\theta) = \psi(\theta/\alpha)$ and the corresponding domain of ψ is

$$\theta(-g^3) \geqslant \theta \geqslant \theta(-g^2) \tag{3.20b}$$

while $G_2[\psi]$ the domain of ψ is

$$\theta(-g^3) \leqslant \theta \leqslant \theta(-g) \tag{3.20c}$$

The analysis which led to (3.19) is acceptable within a range of θ, which does not include singularities of $\tilde{h}(\theta)$. According to (2.27), this range is considerably wider than the range (3.20a). It is

$$\theta(-1) < \theta < \theta(-g^{-1}) \tag{3.21}$$

Hence we are allowed to apply (3.19) to the class of $\psi(\theta)$ defined and nonsingular in the interval (3.20a).

C. Matrix Equations

To analyze (3.19), visualize expanding the appropriate functions of θ in some set of basis functions $\psi_k(\theta)$, which are complete and nonsingular in the range (3.20a). To analyze the first term we write

$$\psi(\theta) = \sum_{k=0}^{\infty} \psi^k \psi_k(\theta) \tag{3.22a}$$

To analyze the second term write

$$\psi_k(\theta/\alpha) = \sum_j A_k{}^j \psi_j(\theta) \tag{3.22b}$$

The third term involves the expansion

$$\frac{\psi_k(\alpha^{-2}\tilde{h}(\theta))}{[\tilde{h}'(\theta)]^\tau} = \sum_j B_k{}^j \psi_j(\theta) \tag{3.22c}$$

Then (3.19) may be recast in matrix form using, as a set of expansion coefficients

$$\gamma_{n,k}(\tau) = \gamma_n(\tau, [\psi_k]) \tag{3.23}$$

The equation reads

$$\sum_{kj} \psi^k [\delta_k{}^j \gamma_{n,j} - A_k{}^j \gamma_{n-1,j} - B_k{}^j \gamma_{n-2,j}] = 0$$

Since the ψ^k are completely arbitrary, the coefficient of each ψ^k must be zero. Hence we see

$$\gamma_{n,k} = \sum_j A_k{}^j \gamma_{n-1,j} + B_k{}^j \gamma_{n-2,j} \tag{3.24}$$

A neater form is obtained if we define

$$\gamma'_{n,k} = \sum_j B_k{}^j \gamma_{n-1,j}$$

Then (3.24) can be written as an expression for a two-component vector as

$$\begin{pmatrix} \gamma_n \\ \gamma'_n \end{pmatrix} = H(\tau) \begin{pmatrix} \gamma_{n-1} \\ \gamma'_{n-1} \end{pmatrix} \tag{3.25}$$

where

$$H_k{}^j(\tau) = \begin{vmatrix} A_k{}^j & \delta_k{}^j \\ B_k{}^j(\tau) & 0 \end{vmatrix} \tag{3.26}$$

Notice that $H_k{}^j$ is a known matrix, once α and $\tilde{h}(\theta)$ are known. For example, if $\psi_k(\theta) = \theta^k$, then

$$A_k{}^j = \alpha^{-k} \delta_k{}^j \tag{3.27a}$$

and $B_k{}^j$ is defined by

$$\frac{[(1/\alpha^2)\, \tilde{h}(\theta)]^k}{[\tilde{h}'(\theta)]^\tau} = \sum_j B_k{}^j \theta^j \tag{3.27b}$$

It may well be that a power series like (3.27b) does not really converge in the entire region of θ. Then one should take all sums to have a finite number of terms and interpret them as fits to the functions in question.

The next step is to interpret (3.25) as an eigenvalue equation. Assume that for large n and almost any ψ

$$\gamma_n(\tau, [\psi]) \to \lambda^n \gamma(\tau, [\psi]) \tag{3.28}$$

and, in particular

$$\gamma_{n,k}(\tau) \to \lambda^n \gamma_k(\tau) \tag{3.29}$$

Then (3.25) becomes the eigenvalue statement

$$\lambda(\tau) \begin{pmatrix} \gamma \\ \gamma' \end{pmatrix} = H(\tau) \begin{pmatrix} \gamma \\ \gamma' \end{pmatrix} \tag{3.30}$$

If $\lambda(\tau)$ is the largest eigenvalue of (3.30), one can return to (3.1) and (3.10) and thereby find

$$q(\tau) = \frac{\tau \ln |\alpha| + \ln \lambda(\tau)}{-\ln |g|} \tag{3.31}$$

In this way, we have constructed a renormalization group calculation of $q(\tau)$.

D. Functional Equations

Under most circumstances one can equally well find eigenvalues from the right- or left-hand form of eigenvalue equations. Thus it is plausible that one can also find $\lambda(\tau)$ from the alternative eigenvalue equation

$$\psi^k \psi'^k \lambda(\tau) = \sum_j \psi^j \psi'^j H_j{}^k(\tau) \tag{3.32}$$

Take (3.22), multiply on the right by $\psi_k(\theta)$, and sum over θ. Define $\psi(\theta)$ by (3.22a), and make use of (3.22b) and (3.22c). The resulting functional equation for ψ is

$$\lambda\psi(\theta) = \psi(\theta/\alpha) + [\psi(\tilde{h}(\theta)/\alpha^2]/[\tilde{h}'(\theta)]^\tau \lambda \tag{3.33}$$

Then λ is the largest eigenvalue of (3.33) defined by the condition that $\psi(\theta)$ should be nonsingular in the range (3.20a).

Equation (3.33) is useful for making analytical progress. The alternative form (3.30) is more useful for numerical work.

4. RESULTS

A. Exact Statements

The exact statements (3.2)–(3.4) imply via (3.31) that $\lambda(\tau)$ obeys

$$\lambda(-\infty) = 1 \tag{4.1a}$$

$$\lambda(-1) = |\alpha| \tag{4.1b}$$

$$\lambda(0) = |g|^{-1} \tag{4.1c}$$

$$\lambda(\tau) \to |\alpha|^{2\tau} \qquad \text{as} \qquad \tau \to \infty \tag{4.1d}$$

To derive (4.1c) notice that (3.33) has, at $\tau = 0$, eigenfunctions $\psi(\theta) = \text{const}$ and, hence, λ obeys

$$\lambda^2 = \lambda + 1$$

The larger solution is (4.1c).

To derive (4.1a), notice that in the range $\theta(-g^2) < \theta < \theta(-g)$, $\tilde{h}'$ obeys $0 < \tilde{h}'(\theta) < 1$. We therefore guess a solution to (3.33) which is, once again, essentially constant. One possible solution, the one which has the largest eigenvalue, is $\lambda = 1$. In this solution, the second term on the right in (3.33) is a small correction.

To get the $\tau \to \infty$ solution neglect the first term on the right in (3.33). The equation then reads

$$\lambda^2\psi(\theta) = \psi\left[\frac{1}{\alpha^2}\tilde{h}(\theta)\right] \Big/ [\tilde{h}'(\theta)]^\tau \tag{4.2}$$

Equation (2.21a) implies the two statements

$$\tilde{h}(\alpha\theta) = \alpha^2\tilde{h}[\alpha^{-1}\tilde{h}(\theta)]$$

$$h'(\alpha\theta) = \tilde{h}'(\theta)\,\tilde{h}'[\alpha^{-1}\tilde{h}(\theta)]$$

A formal solution to (4.2) may be derived as

$$\psi(\theta) = \frac{[\tilde{h}(\alpha\theta)]^p}{[\tilde{h}'(\alpha\theta)]^\tau}$$
$$\lambda = |\alpha|^{-p} \tag{4.3}$$

But we know that as $\theta \to 0$, $\tilde{f}(\theta) = A + B\theta^3$. Xence, if we set $\theta = \theta(-1) - x$ for $x \ll 1$, we have, $\tilde{h}(x) \sim Cx^{1/3}$. Hence, if the solution (4.3) is to be non-singular near $\theta = \tilde{\theta}(-g)$, we must have $p = -2\tau$. Then

$$\lambda = |\alpha|^{2\tau} \tag{4.4}$$

The first term on the right-hand side of (3.33) is relatively negligible in this situation. Thus we have checked (4.1d).

I do not see how to derive (4.1b) from (3.33).

B. Lowest-Order Result

In the very lowest-order analysis, one can replace $\tilde{h}'(\theta)$ in (3.22c) by a constant $\tilde{h}'(\theta) = a^{-1}$. If we then choose the lowest-order expansion function to be $\phi_o(\theta) = 1$, (3.22b) and (3.22c) give

$$A_o{}^j = \delta_o{}^j$$
$$B_o{}^j = \delta_o{}^j a^\tau \tag{4.5}$$

The $k = 0$, $j = 0$ subsector of H thus decouples. In this subsector we have from (3.26)

$$H = \begin{pmatrix} 1 & 1 \\ a^\tau & 0 \end{pmatrix} \tag{4.6}$$

which has at its largest eigenvalue

$$\lambda = (1 + \sqrt{1 + 4a^\tau})/2 \tag{4.7}$$

If $a > 1$, (4.1a) also follows immediately from the lowest-order approximation (4.7). From (4.1d), $a = \alpha^4$ and then (4.1b) implies

$$|\alpha| = (1 + \sqrt{1 + 4\alpha^{-4}})/2$$

which has as its solution $\alpha = -1.2852...$. This compares well with the exact result $\alpha = -1.2886...$. In fact, the approximation (4.7) is always accurate to within a few percent. This lowest-order result can give us some confidence in the overall correctness of our approach.

C. Numerical Results

To obtain numerical results one must first find $\tilde{h}(\theta)$. Using exactly the same method as outlined in Ref. 4, one can find $\tilde{f}(\theta_j) = \tilde{f}_j$, the f value at the points $\theta_j = (j/M)^{1/3}$ where $j = 0, 1, 2, \ldots, M$. As in the earlier reference we solved for $\tilde{f}_j$ at $M = 11$. Then this 12-point fit for the derivative $d\tilde{f}_j$ was obtained by using a 12-term polynomial fit in θ_j^3 and differentiating the polynomial term by term.

Next a 12-point fit to $\tilde{h}$ was obtained by writing $\theta_j = \tilde{h}(\tilde{f}_j) = \tilde{h}_j$ and a similar fit was obtained to

$$[d\tilde{h}(x)/dx]_{x=f_j} = 1/df_j \tag{4.8}$$

An approximate α value was obtained as $\alpha = \tilde{h}_M^{-1}$.

As a result, we could start from good approximations for $\tilde{h}_j$ at the arguments $\tilde{f}_j$. The lowest argument was $\tilde{f}_o = 0$. The highest was $\tilde{f}_M = 1$. Using the theory of Ref. 4, these could be interpreted, respectively, as

$$\begin{aligned} \tilde{f}_o &= \tilde{\theta}(0) = 0 \\ \tilde{f}_M &= \tilde{\theta}(-g) = 1 \end{aligned} \tag{4.9}$$

Hence, the highest j value lies precisely at the right-hand end of the fitting interval (3.20a). The lowest j value is lower than the lower end of the interval.

The first three j values lie outside of the required interval. Hence, these values of $\tilde{h}_j$ were discarded and replaced by the exact value of $\tilde{h}(\theta)$ at the left-hand end

$$\begin{aligned} \theta_3 &= \theta(-g^2) = \alpha^{-2} \\ \tilde{h}_3 &= \tilde{h}(\theta_3) = \theta(-g) = \alpha^{-1} \\ \frac{d\tilde{h}}{d\theta}(\theta_3) &= \alpha^{-2} = \left(\frac{d\tilde{h}}{d\theta}\right)_3 \end{aligned} \tag{4.10}$$

Now we have 10 points in $\tilde{h}(\theta)$ and $d\tilde{h}/d\theta$ in the required interval.

These functions were then fit at N equally spaced theta values by interpolation in a 10-term polynomial expansion. Then (3.23) were used to obtain $N \times N$ matrices $A_k{}^j$ and $B_k{}^j(\tau)$. Finally, $\lambda(\tau)$ was obtained as the largest eigenvalue of $H(\tau)$.

To check this work, use the exact statements of Section 3A or 4A. Table I summarizes this comparison with exact statements. The first row is the "exact" data, in error only because of the error in the determination of α used in columns 2 and 5. The errors are as large as 0.1 for $N = 2$ or 3 and fall to 10^{-4} or 10^{-5} at $N = 8$ and 11.

Table I. Comparison Between Exact Results and Numerical Calculations Employing $N \times N$ Matrices

	$\tau \to \infty$ $q(\tau)/\tau$	$\tau = 0$ $q(\tau)$	$\tau = -1$ $q(\tau)$	$\tau \to -\infty$ $q(\tau)/\tau$
Exact	1.5806	1	0	0.52687
$N=2$	1.5806	1	0.10412	0.53058
$N=3$	1.6821	1	0.00806	0.52737
$N=4$	1.7481	1	0.00414	0.52654
$N=5$	1.5805	1	0.00064	0.52679
$N=8$	1.5805	1	0.00005	0.52691
$N=11$	1.5805	1	0.00001	0.52692
$N=14$	1.5811	1	0.00001	0.52692

I consider this table to be a substantial argument that the method is working satisfactorily.

Using the higher-order results I estimate that the errors in the lowest-order approximation of Section 4B are at the 1 or 2% level. Hence, for most practical purposes, this lowest-order approximation will give reasonable results.

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